Journal of Pure and Applied Algebra 33 (1984) 49-58
North-Holland

QUILLEN DECOMPOSITION FOR SUPPORTS OF EQUIVARIANT COHOMOLOGY WITH LOCAL COEFFICIENTS

Stefan JACKOWSKI
Institute of Mathematics, University of Warsaw, Warsaw, Poland

Communicated by K.W. Gruenberg
Received 9 May 1983
Revised 8 September 1983

Let k be a field of characteristic p. For a topological group G and a $k[\pi_0 G]$-module A we consider the equivariant Borel cohomology theory with local coefficients in A. We denote its value on a G-space X by $H_G^*(X; A)$. This is a module over the even-dimensional part of the k-cohomology ring of the classifying space BG. This ring we denote by H_G for short. We shall investigate the support $V_G(X; A)$ of a module $H_G^*(X; A)$ in the spectrum of the ring H_G.

The support $V_G(X; A)$ has a nice description in the case when A is not only a $k[\pi_0 G]$-module but also a $k[\pi_0 G]$-algebra. Then, for an arbitrary subgroup K of G having fixed points on X, the restriction homomorphism $H_G \to H_K$ induces a map $V_K(A) = V_K(\mathrm{pt}; A) \to V_G(X; A)$. Denoting by $EA(G, X)$ the category whose objects are elementary abelian p-subgroups of G having fixed points on X and whose morphisms are restrictions of inner automrophisms of G, we obtain a natural map

$$\lim \mathrm{ind}_{EA(G,X)} V_E(A) \xrightarrow{\alpha} V_G(X; A).$$

The main theorem of the paper says that α is a homeomorphism of spaces with Zariski topology; when G is a compact Lie group or a discrete group of finite p-cohomological virtual dimension with only finitely many conjugacy classes of elementary abelian p-subgroups, X is either finite-dimensional or compact, and the action satisfies certain local and finiteness conditions.

For finite groups and one-point space our result reduces to the stratification theorem of Avrunin and Scott [2, Theorem 3.3]. Their proof is based on the Serre theorem on cohomology of p-groups (Serre [10]). Our proof uses topological arguments in the spirit of Quillen's work on the spectrum of an equivariant cohomology ring with constant coefficients (Quillen [9]). The equivariant cohomology with local coefficients was used by the author to prove a periodicity theorem for group cohomology with coefficients in a module over a group ring (Jackowski [8]). From it the simplest version of the main result was derived [8, Theorem 3.1]. An algebraic proof of it, also using Serre's theorem, was given by

0022-4049/84/$3.00

Chouinard [6]. We conclude the paper giving a topological proof of the Serre theorem.

1. Annihilators of equivariant cohomology with local coefficients

Let G be a topological group acting on the space X. All topological spaces are assumed to be paracompact and Hausdorff. For any $k[\pi_0 G]$-module A we denote

$$H_G^*(X; A) := H^*(EG \times_G X; p^*A)$$

where p^*A is a local coefficient system on the total space induced by the projection map $p: EG \times_G X \to BG$. The cohomology with local coefficients is here the cohomology of a space with coefficients in the sheaf defined by the local coefficient system (cf. Spanier [11, Ex. 6.F]). All facts about the sheaf cohomology which are used in this paper are collected in [9, Appendix A].

The pairing $k \otimes A \to A$ defines on $H_G^*(X; A)$ the structure of an H_G-module. We shall investigate the annihilator of an H_G-module $H_G^*(X; A)$ which will be denoted by $\mathrm{Ann}_G(X; A)$. To obtain a decomposition theorem we have to impose on a G-space X some additional conditions:

(C) The stalks of the Leray sheaf on the orbit space X/G associated to the presheaf $U \to H_G^*(q^{-1}U; A)$, where $q: X \to X/G$ is the projection map, are naturally isomorphic to the equivariant cohomology of the orbits $H_G^*(q^{-1}\{y\}; A)$.

(F) The orbit space X/G has finite cohomological dimension over k.

1.1. Theorem. *For any G-space X satisfying the conditions* (C) *and* (F) *and for any $k[\pi_0 G]$-module A,*

$$\bigcap_{x \in X} \mathrm{Ann}_G(Gx; A) \subset \mathrm{rad}(\mathrm{Ann}_G(X; A)),$$

where Gx is the orbit of the point $x \in X$.

Proof. We apply the Leray spectral sequence $\{E_r^{pq}(A)\}$ of the map $EG \times_G X \to X/G$. The second term of the spectral sequence is given by

$$E_2^{pq}(A) = \check{H}^p(X/G; \mathscr{H}_G^q)$$

where $\mathscr{H}_G^q$ denotes a sheaf defined in (C). The infinite term is a graded group associated with a filtration $\{F_i\}$ of $H_G^*(X; A)$:

$$E_\infty^{p,n-p}(A) = F_{p-1}^{(n)} F/_p^{(n)}.$$

The Learay spectral sequence $\{E_r^{**}(A)\}$ is a sequence of H_G-modules. The module structure is defined by the pairing $k \otimes A \to A$ and the resulting pairing of the Leray spectral sequences $E_r^{pq}(k) \otimes E_r^{st}(A) \to E_r^{p+s,q+t}(A)$. The finiteness assumption (F) implies that $H^p(X/G; \mathscr{H}_G^q)=0$ for $p>\mathrm{cd}_k(X/G)$. Therefore the filtration $\{F_p\}$ is finite.

Assume $u\in \mathrm{Ann}_G(Gx; A)$ for all $x\in X$ and is homogeneous. Then multiplication by u annihilates stalks of the Leray sheaf. Consequently it annihilates all E_r-terms for $r\geq 2$. Vanishing of $u\cup : E_\infty^{p,*}(A)\to E_\infty^{p,*}(A)$ implies that multiplication by u on $H_G^*(X; A)$ maps F_p into F_{p+1}. If $d>\mathrm{cd}_k(X/G)$, we obtain $u^d\in \mathrm{Ann}_G(X; A)$. Hence $u\in \mathrm{rad}(\mathrm{Ann}_G(X; A))$. □

We shall examine the kernel of the edge homomorphism of the Leray spectral sequence $\{E_r^{**}(A)\}$

$$H_G^*(X; A)\xrightarrow{\bar{\alpha}} \check{H}^0(X/G; \mathscr{H}_G^*).$$

The edge homomorphism is induced by the inclusions of orbits $Gx\subset X$ for all $x\in X$. If X has locally finite orbit structure, then the edge homomorphism has values in the product $\prod_{x\in X} H_G^*(Gx; A)$ (cf. [9, 5.11]). Hence its kernel is the intersection of kernels of restriction homomorphisms to orbits occurring on X.

1.2. Theorem. *For a given G-space X satisfying condition* (F) *and a* $k[\pi_0 G]$-*module A there is an integer n such that for any* $u\in\ker\bar{\alpha}$ *its n-th power* $u^n\in H_G^*(X; \underbrace{A\otimes\cdots\otimes A}_{n\text{ times}})$ *vanishes.* □

The proof is similar to the one of Theorem 2.1 and it uses the pairing

$$E_r^{**}(A)\otimes E_r^{**}(A)\to E_r^{**}(A\otimes A).$$

Assume now that A is a $k[\pi_0 G]$-algebra with unit. Then we have an embedding $j: k\to A$ given by $j(r)=r\cdot \mathbb{1}_A$. The map j induces a homomorphism $j(X): H_G\to H_G^*(X; A)$. The following lemma is obvious:

1.3. Lemma. $\mathrm{Ann}_G(X; A)=\ker j(X)$. □

The last lemma implies that for any subgroup $K\subset G$ and any G-subspace $Y\subset X$ the restriction homomorphism $H_G\to H_K$ maps $\mathrm{Ann}_G(X; A)$ into $\mathrm{Ann}_K(Y; A)$. It gives the following corollary of Theorem 1.1.

1.4. Corollary. *Let A be a* $k[\pi_0 G]$-*algebra. If a G-space X satisfies conditions* (C) *and* (F) *and if only finitely many orbit types occur on X, then*

$$\bigcap_{x\in X} \mathrm{rad}(\mathrm{Ann}_G(Gx; A))=\mathrm{rad}(\mathrm{Ann}_G(X; A)). \qquad \square$$

If A is a $k[\pi_0 G]$-algebra, then $H_G^*(X; A)$ is a non-commutative ring. Hence, if $u\in\ker\bar{\alpha}$, then by 1.2 it is nilpotent.

Remark. The Mayer–Vietoris argument gives proofs of Theorems 1.1 and 1.2 for compact Lie groups acting on compact, not necessarily finite-dimensional, spaces (cf. [9, proof of 3.2]). Thus all following corollaries remain true respectively.

2. The decomposition theorem for G-spaces with elementary abelian isotropy subgroups

From now on we assume that M is a $k[\pi_0 G]$-algebra. Let K be a subgroup of G. The map induced by the restriction homomorphism $H_G \to H_K$ on corresponding spectra we denote by $t_K^G : V_K(k) \to V_G(k)$. We need the following lemma mentioned in the introduction.

2.1. Lemma. *If a subgroup K has a fixed point on a G-space X, then t_K^G maps $V_K(M)$ into $V_G(X; M)$.*

Proof. For any equivariant map $f : G/K \to X$ we have the commutative diagram

$$\begin{array}{ccc} H_G & \longrightarrow & H_G^*(X; M) \\ \downarrow & & \searrow f^* \\ H_K & \longrightarrow & H_K^*(\mathrm{pt}; M) = H_G^*(G/K; M) \end{array}$$

Now the lemma follows from Lemma 1.3. □

Let K and K' be subgroups of G and let $\varphi_g : K \to K'$ be a restriction of an inner automorphism of G defined by an element $g \in G$. Then φ_g induces a map of supports $\varphi_{g*} : V_K(M) \to V_{K'}(M)$; and the following diagram is commutative:

$$\begin{array}{ccc} V_K(M) & & \\ \varphi_{g*}\downarrow & \searrow & \\ & & V_G(X; M) \\ & \nearrow & \\ V_{K'}(M) & & \end{array}$$

The induced map φ_{g*} will be also denoted by $t_{K'}^K$.

For a given set F of subgroups of a group G we denote by $F(G, X)$ the category of subgroups belonging to F and having fixed points on X. The morphisms in the category $F(G, X)$ are restrictions of inner automorphisms of the group G.

Lemma 2.1 and the above remarks enable us to define a map

$$\lim \operatorname{ind}_{F(G, X)} V_E(M) \xrightarrow{\alpha} V_G(X; M).$$

The most important set of subgroups is for us the set of all elementary abelian p-subgroups denoted by EA. We proceed towards the proof of the following:

2.2. Theorem. *The map* $\lim \operatorname{ind}_{EA(G;\, X)} V_E(M) \xrightarrow{\alpha} V_G(X; M)$ *is a homeomorphism of spaces with Zariski topology when either one of the following conditions hold:*

(a) *G is a compact Lie group acting on a space X which is either compact or* $\mathrm{cd}_p(X) < \infty$ *and only finitely many G-orbit types occur on X.*

(b) *G is a discrete group which has only finitely many conjugacy classes of elementary abelian p-subgroups and it contains a normal subgroup G′ of finite index such that* $\mathrm{cd}_p(G') < \infty$. *Moreover G acts on X satisfying conditions* (C) *and* (F) *of Section* 2; *for every point* $x \in X$ *the canonical map* $G/G_x \to X$ *is an embedding onto a subset Gx and only finitely many G-orbit types occur on X.*

Remarks. In the case (a) condition (C) is always fulfilled and orbits are embedded. If $\mathrm{cd}_p(X) < \infty$, then (F) also holds (cf. [9, A.11]).

If the subgroup G' mentioned in (b) has finite-dimensional k-cohomology, then G must have finitely many conjugacy clases of elementary abelian p-subgroups (cf. [9, 14.5]).

First we prove Theorem 2.2 for G-spaces on which all isotropy subgroups are elementary abelian. The following lemma was inspired by Lemma 1.3 of Avrunin and Scott [2].

2.3. Lemma. *For any subgroup E′ of an elementary abelian p-group E and any* $k[E]$-*module A,* $(t_{E'}^{E})^{-1} V_E(A) \subset V_{E'}(A)$.

Proof. We have to prove that if $\mathfrak{p} \notin V_{E'}(A)$, then $t_{E'}^{E}(\mathfrak{p}) \notin V_E(A)$: i.e. $H^*(E'; A)_{\mathfrak{p}} = 0$ implies $H^*(E; A)_{\bar{\mathfrak{p}}} = 0$ where $\bar{\mathfrak{p}} = t_{E'}^{E}(\mathfrak{p})$. The subscript denotes the localization of the group cohomology with coefficients in A with respect to the prime ideal. The extension $E' \to E \to E/E' = Q$ leads to the Hochschild–Serre spectral sequence

$$E_2^{pq}(A) = H^p(Q; H^q(E'; A)) \Rightarrow H^{p+q}(E; A).$$

There is a natural pairing of the latter spectral sequence with the degenerate spectral sequence with constant coefficients: $E_r^{**}(k) \otimes E_r^{**}(A) \to E_r^{**}(A)$. This pairing converts the spectral sequence $\{E_r^{**}(A)\}$ into a sequence of H_E-modules. The projection $E \to E'$ gives a split embedding of the ring $E_2^{0\mathrm{ev}}(k) = H_{E'}$ into H_E. Hence it is enough to prove that $H^*(E; A)_{\mathfrak{p}} = 0$. The $H_{E'}$-module structure on E_2-term is defined by the natural multiplication

$$H^r(E'; k) \otimes H^p(Q; H^q(E'; A)) \to H^p(Q; H^{r+q}(E'; A)).$$

Localization is an exact functor. Therefore, upon localizing the spectral sequence $\{E_r^{**}(A)\}$ with respect to the prime ideal, we obtain a spectral sequence $\{E_r^{**}(A)_{\mathfrak{p}}\} \Rightarrow H^*(E; A)_{\mathfrak{p}}$. It is also clear that $H^p(Q; H(E'; A))_{\mathfrak{p}} = H^p(Q; H^*(E'; A)_{\mathfrak{p}}) = 0$. Therefore $H^*(E; A)_{\mathfrak{p}} = 0$. □

Remark. One can also prove that the edge homomorphism $H^*(E;A)\to E_2^{0*}=H^*(E';A)^Q$ is an epimorphism (cf. Betley [3]).

2.4. Corollary. *If M is a $k[E]$-algebra, then $(t_{E'}^{E})^{-1}V_E(M)=V_{E'}(M)$.* □

Using Corollaries 2.4 and 3.5 we obtain a new proof of the Avrunin–Scott lemma mentioned above.

2.5. Lemma. *Theorem 2.2 is true for G-spaces with elementary abelian isotropy subgroups.*

Proof. Corollary 1.4 implies the following decomposition of the support $V_G(X;M)$:

$$V_G(X;M)=\bigcup_{x\in X} V_G(Gx;M).$$

Assumptions (a) and (b) imply that $H_G^*(Gx;M)=H_{G_x}^*(\mathrm{pt};M)=H^*(BG_x;M)$. The H_G-module structure on $H_G^*(Gx;M)$ is defined by the composition $H_G\to H_{G_x}\xrightarrow{j} H^*(BG_x;M)$; hence $H_G/\mathrm{Ann}_G(Gx;M)\subset H_{G_x}/\mathrm{Ann}_{G_x}(M)$.

Observe that in our case H_{G_x} is a finite H_G-module. For compact Lie groups this is Corollary 2.4 in [9].

In case G satisfies assumption (b), as $\mathrm{cd}_p(G')<\infty$, G' has no p-torsion elements, hence the composition $G_x\subset G\to G/G'$ is injective. Thus the proof is reduced to the finite group case.

The Cohen–Seidenberg theorem implies now surjectivity of the map $V_{G_x}(M)\to V_G(Gx;M)$. Hence α is also surjective.

To prove injectivity of α it is enough to show that if $\mathfrak{p}'\in V_{E'}(M)$, $\mathfrak{p}''\in V_{E''}(M)$ and $t_{E'}^{G}(\mathfrak{p}')=t_{E''}^{G}(\mathfrak{p}'')$, then there is a subgroup E and an ideal $\mathfrak{p}\in V_E(M)$ such that $t_E^{E'}(\mathfrak{p})=\mathfrak{p}'$ and $t_E^{E''}(\mathfrak{p})=\mathfrak{p}''$. The existence of E and $\mathfrak{p}\in V_E(k)$ satisfying those conditions follows from Quillen's theorems [9]. Corollary 2.4 ensures us that in fact $\mathfrak{p}\in V_E(M)\subset V_E(k)$.

As G has only finitely many conjugacy classes of elementary abelian p-subgroups [9, 6.3], it is easy to see that α is a closed map. Hence it is a homeomorphism. □

3. The decomposition theorem for arbitrary G-spaces

Let G be a compact Lie group. Choose an embedding of G into a unitary group $U=U(n)$ and let S be the subgroup of elements of order dividing p in a maximal torus of U. Denote $F=U/S$ the p-flag manifold. For any G-space X the projection $X\times F\to X$ defines the fibration $EG\times_G(X\times F)\to EX\times_G X$ with fiber F. The fiber in the latter fibration is totally non-homologous to zero [9, 6.5]. To compute the equivariant cohomology of $X\times F$ with coefficients in M we need the Leray–Hirsch theorem with local coefficients.

Let $p:E\to B$ be a locally trivial bundle whose fiber is totally non-homologous to zero over a field k. Let Γ be any local system of k-modules on B. Let $\theta : H^*(F;k)\to H^*(E;k)$ denote a cohomological extension of the fiber.

3.1. Theorem. *The map* $\Phi : H^*(B;\Gamma)\otimes H^*(F;k)\to H^*(E;p^*\Gamma)$ *given by* $\Phi(b\otimes f)=\theta(f)\cup p^*(b)$ *is an isomorphism of* $H^*(B;k)$*-modules; and the induced homomorphism* $p^*:H^*(B;\Gamma)\to H^*(E;p^*\Gamma)$ *is a monomorphism.*

Proof. The theorem follows from the Serre spectral sequence for the map p with local coefficients. □

3.2. Corollary. *There is an isomorphism of* H_G*-modules*

$$H_G^*(X\times F;M)=H_G^*(X;M)\oplus\cdots\oplus H_G^*(X;M)$$

and the projection $X\times F\to X$ *induces a monomorphism of equivariant cohomology.* □

Proof of Theorem 2.2. Assume first that Condition (a) is fulfilled, i.e. G is a compact Lie group. Subgroups of G which have fixed points on the p-flag manifold F are precisely elementary abelian p-subgroups of G. Hence all isotropy subgroups on $X\times F$ are elementary abelian and $EA(G,X)=EA(G,X\times F)$. Corollary 3.2 gives us the equality of supports $V_G(X;M)=V_G(X\times F;M)$. The assertion follows now from the commutative diagram of homeomorphisms:

$$\begin{array}{ccc}\lim \operatorname{ind}_{EA(G,X)} V_E(M) & \xrightarrow{\ \alpha\ } & V_G(X;M)\\ \Big\| \approx & & \Big\| \approx\\ \lim \operatorname{ind}_{EA(G,X\times F)} V_E(M) & \xrightarrow{\ \alpha\ } & V_G(X\times F;M)\end{array}$$

Now assume that Condition (b) is fulfilled. Let G' be a normal subgroup of finite index in G whose p-cohomological dimension is finite. The Serre construction [9, 15.9] gives us a G-space Y with finite isotropy subgroups on which G' acts freely. The G-space Y fulfills Condition (b). Embedding the finite quotient group G/G' in the unitary group we obtain a G-action on the flag manifold F. The diagonal G-action on $Y\times F$ also fulfills Condition (b). The subgroups of G which have fixed points on $Y\times F$ are again precisely elementary abelian subgroups of G and for any G-space X, $EA(G,X\times Y\times F)=EA(G,X)$. As Y is contractible, we have an isomorphism

$$H_G^*(X\times Y\times F;M)=H_G^*(X\times F;M)$$

and consequently $V_G(X\times Y\times F;M)=V_G(X;M)$. To complete the proof we proceed as in the case of compact Lie group. □

3.3. Corollary. *Let G be a group satisfying assumptions* (a) *or* (b) *of Theorem* 2.2. *Then the map*

$$\lim \operatorname{ind}_{EA(G)} V_E(M) \to V_G(M)$$

is a homeomorphism. □

3.4. Corollary. *For any G-space X satsifying assumptions* (a) *or* (b) *of Theorem* 2.2

$$\operatorname{rad} \operatorname{Ann}_G(X; M) = \bigcap_{EA(G,X)} \operatorname{Rad} \operatorname{Ann}_G(G/E; M).$$

Proof. The corollary follows from Corollaries 1.4 and 3.2. We proceed exactly as in the proof of Theorem 2.2. □

The last corollaries imply results of Avrunin and Scott [2] and Avrunin [1]. To see that, we introduce for any finite group G acting on a space X and any $k[G]$-module A a new H_G-module $M_G(X; A) = \bigoplus H_G^*(X: L \otimes A)$, where L ranges over the isomorphism classes of indecomposable $k[G]$-modules. The following proposition in the case $X = \mathrm{pt}$ is mentioned in [1] and [5].

For a given $k[G]$-module A, we denote by $\operatorname{End}_k(A)$ the $k[G]$-algebra of k-endomorphisms of A with diagonal G-action.

3.5. Proposition. $\operatorname{Ann}(M_G(X; A)) = \operatorname{Ann}_G(X; \operatorname{End}_k(A))$.

Proof. The G-isomorphism $\operatorname{End}_k(A) = A^* \otimes A$ gives an inclusion $\operatorname{Ann}(M_G(X; A)) \subset \operatorname{Ann}_G(X; \operatorname{End}_k(A))$. To prove the converse inclusion we consider the G-pairing $L \otimes A \otimes \operatorname{End}_k(A) \to L \otimes A$ defined by evaluation. This pairing converts $H_G^*(X; L \otimes A)$ into a unitary $H_G^*(X; \operatorname{End}_k(A))$-module. If $u \in \operatorname{Ann}_G(X; \operatorname{End}_k(A))$, then for any $z \in H_G^*(X; L \otimes A)$, $u \cdot z = u \cdot (\mathbb{1} z) = (u \cdot \mathbb{1}) z = 0$. □

3.6. Corollary. $\operatorname{supp}(M_G(X; A)) = V_G(X; \operatorname{End}_k(A))$. □

Now the main theorem of Avrunin [1] follows from Corolary 3.4 applied to the one-point space and finite groups, together with Proposition 3.5. Theorem 3.3 of Avrunin and Scott [2] follows from Corollary 3.3 applied to finite groups and Corollary 3.6.

Following Quillen [9], Avrunin and Scott [2] stated their stratification theorem in various forms (cf. [2, 3.1–3.4]). The proofs of Quillen and Avrunin–Scott carry over verbatim to give analogous versions of our Theorem 2.2.

We finish this section by considering elements of equivariant cohomology restricting trivially to all elementary abelian subgroups of isotropy groups.

3.7. Theorem. *Let X be a G-space satisfying assumption* (a) *or* (b) *of Theorem* 2.2. *For any* $k[\pi_0 G]$*-module A there exists an integer n such that for any*

$$u \in \bigcap_{E \in EA(G,X)} \ker\{H_G^*(X; \mathrm{A}) \to H_E^*(\mathrm{pt}; A)\}$$

$u^n = 0$ *in* $H_G^*(X; A \otimes \cdots \otimes A)$.

Proof. Follows from Theorem 1.2, and the embedding of the equivariant cohomology of X into the equivariant cohomology of a space with elementary abelian isotropy groups provided by Corollary 3.2. □

3.8. Corollary. *If M is a $k[\pi_0 G]$-algebra, then any element of $H^*_G(X; M)$ which restricts trivially to all elementary abelian subgroups is nilpotent.* □

For $X = \mathrm{pt}$, G finite and $M = \mathrm{End}_k(A)$, the last corollary was proved by Carlson [4].

4. The Serre Theorem

The purely algebraic approach to the problems discussed in this paper is based on the following theorem of J.-P. Serre [10]. We give here a topological proof of it.

4.1. Theorem. *Let G be a finite p-group which is not elementary abelian. Let $k = \mathbb{Z}/p\mathbb{Z}$. Thee is a sequence of non-zero elements $y_1, \dots, y_j \in H^1(G;$ k) such that $\beta y_1 \cup \cdots \cup \beta y_j = 0$; where β is the Bockstein homomorphism associated to the short exact sequence*

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0.$$

Proof. Let $G_1, \dots, G_n$ be the complete set of maximal proper subgroups of G. Let $y_i \in \mathrm{Hom}(G, k) = H^1(G; k)$ be an element associated with subgroup G_i. The cohomology class $z = \beta y_1 \cup \cdots \cup \beta y_n$ restricts trivially to every proper subgroup of G; hence to every elementary abelian p-subgroup. Thus by the Quillen nilpotency theorem (Corollary 3.8 in the case $M = k$) the class z is nilpotent. □

The Serre theorem also follows easily from the localization theorem for equivariant cohomology (cf. [7, §3.2]).

It is not difficult to check that the Bockstein of an element $y \in H^1(G; k)$ is the mod p reduction of the Euler class of the one-dimensional complex representation $G \xrightarrow{y} k \subset \mathbb{C}^*$. One can also prove that if G is not an elementary abelian group, then there are non-zero two-dimensional integral cohomology classes whose product vanishes.

Acknowledgements

This work was started once in 1980, and now finished, at The Mathematics Institute of Aarhus University. I thank very much for the Institute's hospitality. And I thank Bob Oliver for reading of the manuscript.

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